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Amplitude estimation using three-parameter sine fitting algorithm in the Planck-Balance

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***Abstract* – The Planck-Balance (PB) is a table-top version of a Kibble balance. In contrast to many existing Kibble balances, the coil is moved sinusoidally and an ac rather than a dc signal is generated in the dynamic mode. The three-parameter sine fitting algorithm is applied to estimate the amplitudes of the induced voltage and the coil motion, which are used to determine the force factor *Bl* of the voice coil of the electromagnetic force compensation balance. However, the three-parameter sine fitting algorithm is not robust against some perturbations, e.g. additive Gaussian white noise, quantization error, and frequency error. These effects have influences on the accuracy of the amplitude estimation. Based on numerical simulations and correlation analyses, the effects of these perturbations are determined. By optimizing measurement and data processing approach, the bias and standard deviation of the estimated amplitude could be effectively reduced, and thus the accuracy of the force factor *Bl* in the dynamic mode can be improved.**

Keywords* – *Planck-Balance, three-parameter sine fitting, Monte Carlo simulation

I. INTRODUCTION

After the redefinition of the unit of mass, the kilogram, the Kibble balance is one possible approach to calibrate mass standards in terms of a fixed value of the Planck constant with zero uncertainty [1]. The Planck-Balance (PB) is a table-top sized Kibble balance and is currently under development in a collaboration between the Physikalisch-Technische Bundesanstalt (PTB) and the Technische Universität Ilmenau (TUIL) [2].

The Kibble balance has two measuring modes: static mode and dynamic mode. In the dynamic mode of existing Kibble balance experiments, the coil is usually moved at a constant velocity. In contrast to these Kibble balances, the coil of the PB is sinusoidally moved through the magnetic (B-)field in an oscillatory manner, thus inducing an ac voltage across the coil ends. This voltage is digitized by means of a high-precision digital multimeter (Agilent 3458A). If it is assumed that the coil motion and the induced voltage are perfectly sinusoidal, the force factor $B{\cdot}l$ (l denotes the coil wire length) can be determined by the oscillation frequency, the amplitudes of the coil motion and the induced voltage. In the PB set-up only the amplitudes are required to be estimated since the oscillation frequency can be accurately measured with a frequency counter. The motion of the coil with respect to the magnet is measured with a commercial laser interferometer, which provides the position data as a function of time. The measurements of voltage and position are synchronized by means of an external trigger source with a trigger frequency of 1 kHz.

The three-parameter sine fitting algorithm is applied to determine the amplitudes of the induced voltage and the coil motion. However, in practice, the sampled signal is not an ideal single-component sine wave due to some perturbations like, e.g., additive Gaussian white noise, quantization error or harmonic distortion. The fitting algorithm is not robust against these perturbations, and may result in a bias of the amplitude, which will directly affect the accuracy of Bl. In this paper, the effects of aforementioned perturbations are analyzed by numerical simulations. The bias and standard deviation of the estimated amplitude provided by the three-parameter sine fit are evaluated by the Monte Carlo simulation.

II. DYNAMIC MODE OF PLANCK-BALANCE

In the dynamic mode of the PB, the coil oscillates through the B-field generated by the magnet system. The coil position is measured by using an interferometer, while the induced voltage in the coil is measured by using a voltmeter synchronously. The measured coil motion is assumed to be an ideal single-component sine wave with an amplitude S and an initial phase φ_s:

$$s(t) = S_0 + S\sin(\omega t + \varphi_s). \quad (1)$$

Here S_0 is the dc offset and ω is the angular frequency, $\omega = 2\pi f_{\text{sig}}$, where f_{sig} denotes the oscillation frequency.

The coil velocity can be obtained as the derivative of (1) with respect to the time t as

$$v(t) = \omega S\cos(\omega t + \varphi_s). \quad (2)$$

The induced voltage is also assumed to be a perfect sine wave with an amplitude U and an initial phase φ_u:

$$u(t) = U\cos(\omega t + \varphi_u). \quad (3)$$

Assuming that the force factor Bl is a constant during the whole range of coil movement, Bl can be calculated to divide the amplitude of the induced voltage by the amplitude of coil velocity as

$$Bl = \frac{U}{\omega S} = \frac{U}{2\pi f_{\text{sig}} S}. \quad (4)$$

In the PB system, the oscillation frequency f_{sig} can be accurately measured by a frequency counter. Therefore, the accuracy of Bl depends on the estimation of amplitudes U and S.

III. THREE-PARAMETER SINE FIT

A sinusoidal signal can be described as follows:

$$y(t) = Y_0 + Y\sin(\omega t + \varphi) \quad (5)$$

where Y is the amplitude, φ the initial phase, and Y_0 a dc offset. Equivalently, the signal (5) can be written as a linear combination of two shifted sine waves:

$$y(t) = Y_0 + A\sin(\omega t) + B\cos(\omega t) \quad (6)$$

where A and B are the amplitudes of in-phase and in-quadrature components, respectively. In (6), the signal frequency is known, and thus only three parameters Y_0, A and B are required to be estimated.

When a set of M samples $y_1, y_2, \ldots, y_M$ from a sine wave is sampled at the time instants $t_1, t_2, \ldots, t_M$, a linear least squares method can be used to determine the best sine wave parameters by minimizing the sum of the squares of the following errors:

$$\min_{A,B,Y_0} \sum_{i=1}^{M} (y_i - Y_0 - A\sin(\omega t_i) - B\cos(\omega t_i))^2. \quad (7)$$

The estimated parameters of the sine wave can be calculated in a matrix form:

$$\begin{bmatrix} A \\ B \\ Y_0 \end{bmatrix} = (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} \quad (8)$$

with

$$\mathbf{D} = \begin{bmatrix} \sin(\omega t_1) & \cos(\omega t_1) & 1 \\ \sin(\omega t_2) & \cos(\omega t_2) & 1 \\ \vdots & \vdots & \vdots \\ \sin(\omega t_M) & \cos(\omega t_M) & 1 \end{bmatrix}.$$

The amplitude Y can be determined from $Y = \sqrt{A^2 + B^2}$.

IV. INFLUENCES ON AMPLITUDE ESTIMATION

In order to investigate some effects on the amplitude estimation provided by the three-parameter sine fitting algorithm, numerical experiments are implemented to simulate the induced voltage measurement. Firstly, a single-component sine wave is generated with the amplitude U_{ref} = 0.3051 V, the initial phase φ = -1.5876 rad and the signal frequency f_{sig} = 4 Hz. These parameters are chosen according to real measurement data [4]. A data set $\{u_i\}_{i=1}^{M}$ is generated from the ideal sine wave and taken as the nominal points. For error comparison of amplitude, the value of U_{ref} is taken as the reference value. Then, one or multi-perturbations are superimposed on the nominal points. The three-parameter sine fitting algorithm is implemented to estimate the amplitude $\widehat{U}$. Finally, the relative bias of the amplitude ε_u is calculated, as $\varepsilon_u = |\widehat{U} - U_{\text{ref}}|/U_{\text{ref}}$.

A. Additive Gaussian white noise

The amplitude estimation is affected by additive Gaussian white noise (AGWN) with zero mean and standard deviation σ_n. The theoretical expressions have been derived in [4-5] as follows:

$$\varepsilon_u \approx \frac{\sigma_n^2}{M U_{\text{ref}}^2}, \quad (9)$$

$$\sigma_u = \sqrt{\frac{2}{M}}\sigma_n, \quad (10)$$

where σ_u denotes the standard deviation of the estimated amplitude.

In order to validate the theoretical expressions (9) and (10), the Monte Carlo method (MCM) is used to evaluate the relative bias and standard deviation of the amplitude. The data set $\{u_i\}_{i=1}^{10^4}$ is taken as nominal points, which is generated from a pure sine wave with the sampling frequency f_s = 1 kHz and the sampling time T = 10 s. AGWN with different values of σ_n is superimposed on the nominal points. The three-parameter sine fitting algorithm is used to estimate the amplitude $\widehat{U}$. The process is repeated 10^4 times. The relative bias of the amplitude ε_u is calculated by comparison of the mean value of $\widehat{U}$ with U_{ref}. The standard deviations associated with the estimated amplitude as a function of the noise level are presented in Fig. 1. According to (10), the theoretical values are also calculated with 10^4 samples and different values of σ_n. The solid line represents the theoretical results in Fig. 1.

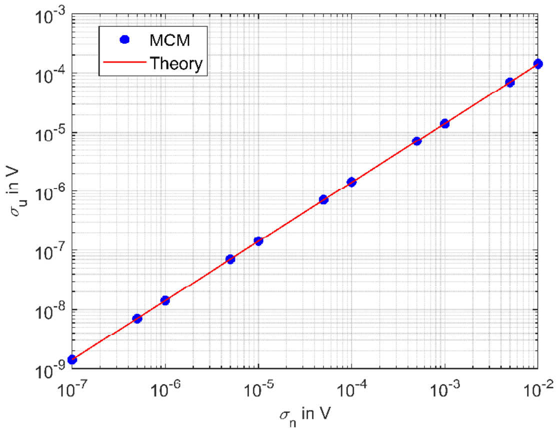

Fig. 1. Standard deviation of the amplitude as a function of the noise standard deviation. The circles represent the simulated values obtained by the Monte Carlo method. The solid line represents the theoretical values given by (10).

From (9) and (10), it can be deduced that the bias is negligible with respect to its standard deviation. Fig. 1 shows a good agreement between theoretical and simulated results. It can be seen that σ_u increases with the increase of σ_n. If $\sigma_n = 10^{-4}$ V, the corresponding σ_u is in the order of 10^{-6} V. The σ_u can be reduced by either reducing σ_n or increasing the number of samples M.

B. Sampling strategy

When measuring the induced voltage and the coil motion, the sampling frequency f_s and the sampling time T need to be determined. The number of samples M is equal to the product of f_s and T. As mentioned in section A, the bias of the estimated amplitude and associated standard deviation can be reduced by increasing M. With a fixed value of T, more samples can be obtained by increasing f_s. However, a higher sampling rate f_s may result in a lower amplitude resolution of the digitizer with q bits of resolution, which results in an increased quantization error.

Fig. 2 gives an example of the influence of the sampling frequency on the amplitude estimation. The quantization error is generated and superimposed on the nominal points. It is assumed that the quantization error is uniformly distributed with the variance $\sigma_q = FSR/(\sqrt{12} \cdot 2^q)$, where FSR is full-scale range [6]. The measurement time is $T = 10$ s, and the corresponding bits are 21 and 18 for the sampling frequencies $f_s = 1$ kHz and 10 kHz, respectively. The MCM with 10^4 trials is implemented to evaluate the standard deviation of the amplitude. The results show that when the sampling frequency increases by a factor of ten from 1 kHz to 10 kHz, the corresponding standard deviation of the amplitude increases by a factor of about 2.5. However, the value of σ_u is in the order of 10^{-9}, which is much lower than the noise level. Therefore, the quantization error is negligible in this case.

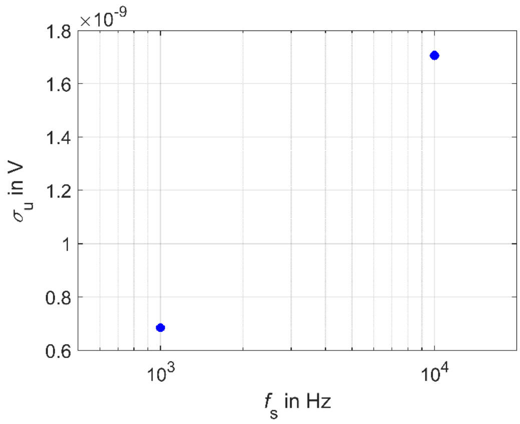

Fig. 2. Standard deviation of the amplitude as a function of sampling frequency by using the Monte Carlo simulation.

If $f_s = 10$ kHz is used instead of 1 kHz and the noise standard deviation of $\sigma_n = 10^{-4}$ V, the corresponding σ_u is reduced by an order of magnitude according to (10). Therefore, $f_s = 10$ kHz can be adopted to improve the accuracy of amplitude estimation.

The signal frequency f_{sig}, the sampling frequency f_s and the number of samples M satisfy the relation:

$$\frac{f_{sig}}{f_s} = \frac{J+\delta}{M}, \tag{11}$$

where J and δ are respectively the integer and the fractional parts of the number of sine wave cycles.

When $\delta = 0$, it is called coherent sampling, (9) and (10) can only be valid for this case. However, when non-coherent sampling ($\delta \neq 0$) occurs, the three-parameter sine fit provides a biased amplitude due to harmonic distortion [7]. The details have been given in [8]. Multiharmonic sine fitting or the extraction of integer periods of the sine wave can be used to reduce the bias of the amplitude provided by the three-parameter sine fit.

C. Frequency error

In the three-parameter sine fitting algorithm, the signal frequency f_{sig} is assumed to be known, and it is taken as an input to the fitting model. If there is an error

δf_{sig} of the input frequency, an additional contribution will be included in the fitting model, which is described by a linearly damped oscillation [9].

A numerical simulation has been implemented in order to investigate the effect produced by a frequency error δf_{sig}. The data set $\{u_i\}_{i=1}^{10^4}$ is generated with f_{sig} = 4 Hz. A frequency error δf_{sig} is added to f_{sig}, and the obtained frequency $(\delta f_{\text{sig}} + f_{\text{sig}})$ is taken as the known frequency of the three-parameter sine fit to evaluate the amplitude. Finally, the relative bias of the amplitude is calculated and shown in Fig. 3.

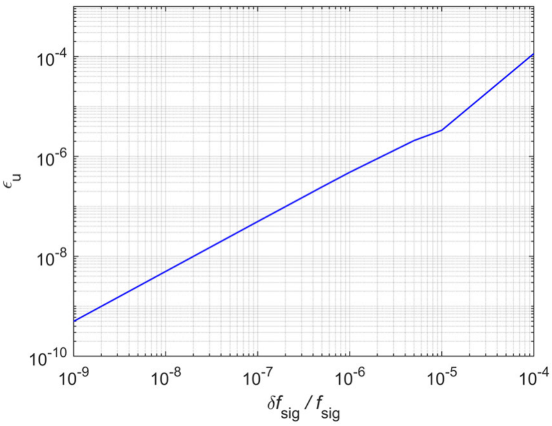

Fig. 3. Relative bias of the amplitude as a function of the signal frequency error.

It can be seen in Fig. 3 that a more accurate amplitude can be obtained with a lower frequency error used in the fitting model. A frequency counter can measure the frequency accurately (< 10^{-8}), which can be used as the input frequency of the model and is sufficient for the accuracies aimed for the PB (about 10^{-8}). Experiments using other algorithms, e.g., the four-parameter sine fit and interpolated DFT method, showed that those algorithms can also reach frequency estimations with errors in the order of 10^{-8}.

V. CONCLUSIONS AND OUTLOOKS

Perturbations in the signal can result in a bias of the amplitude estimation when using the three-parameter sine fitting algorithm. The effects of AGWN, quantization error and frequency error have been investigated by numerical simulations in this paper. The bias and the associated standard deviation due to AGWN depend on the noise level and the number of samples. In order to improve the accuracy of the amplitude estimation, a sampling frequency f_s = 10 kHz can be used instead of 1 kHz without a notable influence of the increased quantization noise. Moreover, the simulation results indicate that the frequency error of the fitting model can also cause a biased amplitude. However, the use of a frequency counter or some effective algorithms can keep this influence negligible.

In the near future, work will be carried out in order to include other important effects, e.g. a nonlinear *B*-field or time jitter. All investigated error sources will be taken into account to evaluate the uncertainty of the force factor *Bl* in the dynamic mode. The aim is to create a good model for the implementation in the virtual Planck-Balance, i.e. a Monte Carlo based uncertainty estimation to the PB [2].

VI. ACKNOWLEDGMENT

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