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Theoretical analysis of the impact of normalizing composite quality indicator components on their extrema

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ABSTRACT
Quality assessment of complex objects commonly employs composite quality indicators constructed from weighted individual quality indicators and management tools. When these variables differ in units or ranges, normalization to dimensionless values in [0, 1] is required. This transformation yields an alternative quality model whose extremum characteristics may differ from those of the original non-normalized model. This issue has received limited attention in the literature. The aim of this study is to analyse the effect of parameter normalization on the extremum of a composite quality indicator. The analysis considers general quality models based on weighted arithmetic, harmonic, geometric, and quadratic mean aggregation functions, each evaluated with and without normalization of individual quality indicators. The results reveal that normalization affects both the location and type of the extremum, as well as the optimal values of quality management tools, for arithmetic, harmonic, and quadratic mean convolutions. Notably, for the weighted geometric mean, the optimal management tool value is independent of normalization, making this convolution particularly suitable for practical quality management. These findings provide a foundation for developing dedicated back-transformation methodologies to ensure accurate real-world application of results obtained from normalized quality models.

Section: RESEARCH PAPER

Keywords: individual quality indicators; composite quality indicator; quality management tools; normalization; extremum analysis

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1. INTRODUCTION

When assessing and predicting the quality of complex systems, qualimetric models are frequently employed in the form of a composite quality indicator, Q.

The corresponding functional is constructed using n individual quality indicators Y_i (where $1 \leq i \leq n$) along with their respective weighting coefficients K_i. In this framework, the weights satisfy the following condition:

$$\sum_{i=1}^{n} K_i = 1\,. \tag{1}$$

Each individual quality indicator Y_i reflects a certain property of the object under evaluation and may depend on m different or identical technical, technological, or organizational management tools X_j (where $1 \leq j \leq m$) [1], [2].

Examples of technical quality management tools may include the control of an object's dimensions or weight, or both; technological tools, on the other hand, encompass parameters such as the speed or operational interaction of its components. An example of organizational tools includes metrics reflecting management impacts on the company's activities, measured in points according to ISO 9004 [3] requirements on an interval scale (difference scale) [4], etc.

Then, the relationship among the specified parameters, Q, Y_i, K_i, and X_j, can be represented by a functional of the generalized form [5], [6]:

$$Q = \varphi(Y_1, \dots, Y_i, \dots, Y_n; K_1, \dots, K_i, \dots, K_n; n), \tag{2}$$

$$Y_i = \phi\left(X_1, \dots, X_j, \dots, X_m; m\right), \tag{3}$$

$$Q = f\left(X_1, \dots, X_j, \dots, X_m; m; K_1, \dots, K_i, \dots, K_n; n\right). \tag{4}$$

Here, φ, ϕ, and f denote the specific functional forms selected by an expert based on the characteristics of the evaluated object [6].

Quite often, functions of form (2) cannot be explicitly expressed. Consequently, when defining Q based on the characteristics of the object, convenience of analysis, or the preferences of the researcher, a specific weighted average is used as an aggregation function [7], [8]:

- arithmetic:

$$Q_{\text{arith}} = \frac{\sum_{i=1}^{n} K_i \cdot Y_i}{\sum_{i=1}^{n} K_i}; \tag{5}$$

- harmonic:

$$Q_{\text{harm}} = \frac{\sum_{i=1}^{n} K_i}{\sum_{i=1}^{n} \frac{K_i}{Y_i}}; \tag{6}$$

- geometric:

$$Q_{\text{geom}} = \left(\prod_{i=1}^{n} Y_i^{K_i} \right)^{1/\sum_{i=1}^{n} K_i}; \tag{7}$$

- quadratic:

$$Q_{\text{quad}} = \sqrt{\frac{\sum_{i=1}^{n} K_i \cdot Y_i^2}{\sum_{i=1}^{n} K_i}}, \tag{8}$$

or other.

The positive or negative relationship between Y_i and X_j depends on the natural properties of a particular object; hence, a corresponding shift in Y_i may lead to both an increase and a decrease in the Q level, as dictated by the specific quality management parameters of the process [6].

As a result, an extremum Q_{ext} may occur, aligning with the target objective of object quality management. This objective is met by identifying optimal management tool levels (X_{jext}), thereby maximizing the rational value Q_{ratio} within the feasible implementation range of Y_i and X_j, and preventing the minimization of Q.

Formally, the X_{jext} definition can be provided by resolving the system of equations [9]:

$$\frac{\partial Q}{\partial X_1} = 0; \dots \dots; \frac{\partial Q}{\partial X_j} = 0; \dots \dots; \frac{\partial Q}{\partial X_m} = 0, \tag{9}$$

and found X_{jext} when substituting into functional (4) will determine the corresponding Q_{ext}.

The type of extremum (maximum – "*max*", or minimum – "*min*") can be determined by one of the known methods [9]:

– by the change of sign when transitioning from the value of the first derivative $Q'_{(x-)}$ (denoted by a prime symbol) to $Q'_{(x+)}$ at levels of X_j slightly smaller and slightly larger than X_{jext}: a change of sign from "plus" to "minus" corresponds to the maximum of Q_{ext}, and from "minus" to "plus" to its minimum; the preservation of the sign indicates the absence of an extremum at the point X_{jext};

– by the sign of the second derivative $Q''_{(x_{ext})}$ (denoted by a double prime) at the point X_{jext}: a maximum for Q_{ext} is expected if $Q''_{(x_{ext})} < 0$, and its minimum if $Q''_{(x_{ext})} > 0$;

– according to direct computational analysis of Q values during the variation of X_j.

If the optimal values X_{jext} cannot be determined or fall outside the limits of practical feasibility, alternative configurations of X_j within their permissible realization range should be established to ensure the highest rational value, Q_{ratio}. Naturally, if the global mathematical maximum $Q_{\max}$ is identified, then $Q_{\text{ratio}} = Q_{\max}$.

It is important to note that both individual quality indicators Y_i and management tools X_j can vary in nature, dimensionality, and range of existence. This requires models (2)–(4) to simultaneously incorporate heterogeneous variables—such as physical mass (in kg), length (in m), or electrical resistance (in ohms)—alongside subjective metrics (e.g., "7 out of 10 points" or "40 out of 100 points" on ratio or absolute scales). Consequently, their disparate scales make direct comparison impossible. Therefore, an essential prerequisite when forming a convolution Q is to ensure identical dimensions and variation ranges for all considered Y_i and X_j parameters.

A common approach to achieving scalability for individual quality indicators is to normalize Y_i relative to their base values, thereby obtaining the corresponding normalized values y_i.

This transformation is typically performed using the following formulas [8]:

$$y_i = \frac{Y_i}{Y_{i\max}}, \tag{10}$$

or

$$\bar{y}_i = \frac{Y_{i\min}}{Y_i}. \tag{11}$$

Here, $Y_{i\max}$ represents the largest actual (base) value of Y_i, which is applied when an increase in the individual indicator drives an increase in Q. Conversely, $Y_{i\min}$ denotes the smallest actual (base) value, used when an increase in Y_i causes a decrease in Q.

Consequently, the normalized parameters are restricted to the range $0 < y_i \leq 1$ a [7], [8].

Similarly, the normalization of X_j can be performed by considering $X_{j\max}$, which represents the largest actual (base) value of X_j whose increase is assumed to drive an increase in Y_i. Conversely, $X_{j\min}$ denotes the smallest actual (base) value, whose increase, owing to the physical nature of the object, causes a decrease in Y_i.

When Y_i *exerts a positive impact on* Q, *and* X_j *similarly exhibits a positive effect on* Y_i, the normalization procedure remains straightforward [6].

When Y_i *exerts a negative impact on* Q, the following scenarios may occur, depending on the nature of the relationship between Y_i and X_j:

- an increase in X_j is accompanied by an increase in Y_i, which necessitates the normalization of X_j according to the formula:

$$x_j = \frac{X_j}{X_{j\max}}. \tag{12}$$

This necessitates the normalization of X_j using formula (12), and that of Y_i using formula (11), which significantly complicates subsequent mathematical transformations;

- an increase in X_j is accompanied by a decrease in Y_i:

$$\bar{x}_j = \frac{X_{j\min}}{X_j}. \tag{13}$$

This requires the normalization of X_j via formula (13) and that of Y_i via formula (10). This approach can simplify the mathematical analytical procedures for determining Q_{ext} and X_{jext}, the deliberate adjustment of which directly regulates the quality of the object [8].

In certain scenarios, selecting an appropriate functional form (3), where $0 < x_j \leq 1$ increases, allows for a *predetermined* (either positive or negative) impact of $0 < Y_i \leq 1$ on Q. Under such conditions, the application of formula (11) becomes redundant.

Naturally, if individual quality indicators Y_i or management tools X_j, share the same dimensions, their normalization becomes optional. However, the tasks of harmonizing these parameters and assessing their impact on Q remain critical, necessitating that the significance and variation ranges of these quantities be accounted for within their common scale of measurement.

2. PROBLEM STATEMENT

Logic suggests that if the relationship between Q and all individual indicators Y_i is monotonic (strictly positive or strictly negative), the occurrence of an extremum Q_{ext} is highly unlikely. In such scenarios of quality management, the values of $X_{j\text{ratio}}$ (or $x_{j\text{ratio}}$) are established within their feasible implementation range to achieve the rational, largest value Q_{ratio} rather than a mathematical extremum [7]. Conversely, an extremum Q_{ext} within a positive relationship between Q and Y_i may arise only if at least one individual quality indicator Y_i (or y_i) exhibits its own extremum during variations in X_j (or x_j). If the system comprises multiple individual indicators ($i > 1$), the emergence of Q_{ext} requires at least two indicators Y to exert opposing effects on Q.

Empirical evidence indicates that functions of forms (2)–(4) are frequently nonlinear [6]. Additional nonlinearity is introduced during parameter normalization via transformations (11) and (13), as well as through the application of weighted harmonic (6), geometric (7), or quadratic (8) means, followed by the determination of Q (or Q_{ratio}) using the nonlinear system (9) [10]. Consequently, accounting for these factors within normalized parameters yields a new quality management model for the same object. This transition may shift the optimal values to X_{jext} (or x_{jext}), which subsequently redefine Q_{ext} or Q_{ratio}, thereby introducing uncertainty into practical quality management decisions.

This specific issue has not been previously addressed in literature.

Therefore, the purpose of this work was a generalized (theoretical) analysis of the impact of normalizing the components of a complex indicator of the quality of an object on the characteristics of its extremum.

Achieving this objective involves a comparative evaluation of x_{2ext} (derived with normalized Y_1 and Y_2) against x_{1ext} (obtained without normalization), or x_{ratio} against $x_{1\text{ratio}}$ under strictly comparable conditions.

3. RESEARCH SECTION

To simplify the theoretical analysis, the following assumptions are made:

The composite quality indicator Q, defined by the general functional form (2), depends on only two individual quality indicators, Y_1 and Y_2 (or their normalized equivalents, y_1 and y_2). Each indicator, in turn, depends on a single common quality management tool X (or its normalized form, x) according to the general functional form (3):

$$Y_1 = \phi_1(X); Y_2 = \phi_2(X)\,, \tag{14}$$

- *for all subsequent computational scenarios*, the selected functional forms ϕ_1, ϕ_2, as well as the normalization of X via formulas (12) or (13), are expressed directly in terms of x:

  $$Y_1 = \phi_1(x); Y_2 = \phi_2(x); \tag{15}$$

  where a monotonic increase in x leads to an increase in Q through the growth of Y_1 (or y_1), whereas an increase in Y_2 (or y_2) causes a decrease in Q;
- according to condition (1), the weighting coefficients satisfy the following relation:

  $$K_1 + K_2 = 1. \tag{16}$$

It should be noted that the designation of Y_1 as the indicator driving an increase in Q and Y_2 as the indicator causing a decrease in Q (along with their normalized counterparts y_1 and y_2 is entirely arbitrary. Each function may possess a distinct mathematical form, range of definition, and maximum value (if applicable). Based on these assumptions, the following analytical results were obtained.

Most frequently, Q is modeled using a weighted arithmetic mean convolution (5) due to its simplicity and the linear relationship between Q and Y_i [6].

In the first stage of the analysis, denoted by the subscript "11" (where the first index indicates the convolution type and the second signifies the **absence of normalization**), the composite quality indicator formulation based on relations (5) and (16) simplifies to:

$$Q_{11\text{arith}} = K_1 \cdot Y_1 + K_2 \cdot Y_2, \tag{17}$$

and its first derivative is expressed as:

$$Q'_{11\text{arith}} = K_1 \cdot Y'_1 + K_2 \cdot Y'_2. \tag{18}$$

Setting this derivative to zero $Q'_{11} = 0$ and performing algebraic rearrangements yields:

$$\frac{Y'_1}{Y'_2} = -\frac{K_2}{K_1}. \tag{19}$$

The parameter value $x_{11\text{arith}.ext}$ corresponding to the potential extremum $Q_{11\text{arith}.ext}$ is implicitly defined by substituting expressions (14) into the left-hand side of equation (19), while the right-hand side represents a constant negative ratio.

The specific *type* of this extremum (maximum or minimum) can be determined either by analyzing the sign change of the first derivative (18) in the neighborhood of $x_{11\text{arith}.ext}$, or by evaluating the sign of the second derivative [9]:

$$Q_{\text{arith}}^{11} = K_1 \cdot Y''_{11} + K_2 \cdot Y''_2, \tag{20}$$

which depends on the functional forms of Y_1, Y_2 defined by equations (14) or (3) and the values of K_1 and K_2.

If no mathematical extremum exists, the rational level $Q_{11\text{arith}.ext}$, and its corresponding argument $x_{11\text{arith.ratio}}$ are determined through direct computational evaluation of equation (17) across the feasible range of x.

When individual indicators Y_1 and Y_2 are **normalized** to y_1 and y_2, the baseline aggregation function (5) takes the form:

$$Q_{2\mathrm{arith}} = \frac{\sum_{i=1}^{n} K_i \cdot y_i}{\sum_{i=1}^{n} K_i}. \tag{21}$$

For this scenario (denoted by the subscript "12", indicating a normalized arithmetic mean), equation (17) is modified as follows:

$$Q_{12\mathrm{arith}} = K_1 \cdot y_1 + K_2 \cdot y_2 , \tag{22}$$

which, substituting expressions (21), (16), and (10), yields:

$$Q_{12\mathrm{arith}} = K_1 \cdot \frac{Y_1}{Y_{1\max}} + K_2 \cdot \frac{Y_2}{Y_{2\max}}. \tag{23}$$

Next, applying a similar procedure to relations (19) and (20) yields the following equations after algebraic transformations:

$$\frac{Y_1'}{Y_2'} = -\frac{K_2}{K_1} \cdot \frac{Y_{1\max}}{Y_{2\max}}, \tag{24}$$

$$Q_{12\mathrm{arith}}'' = -\frac{K_1}{Y_{1\max}} \cdot Y_1'' + \frac{K_2}{Y_{2\max}} \cdot Y_2'' . \tag{25}$$

The parameter value $x_{12\mathrm{arith}.ext}$ corresponding to the potential extremum $x_{12\mathrm{arith}.ext}$ is implicitly defined by substituting expressions (15) into the left-hand side of equation (24). In this scenario, the right-hand side becomes a negative ratio that quantitatively differs from the value obtained via formula (19); thus, in the general case, $x_{12\mathrm{arith}.ext} \neq x_{11\mathrm{arith}.ext}$.

The appropriate type of $Q_{2\mathrm{arith}.ext}$ extremum (maximum or minimum), assuming a feasible value of $x_{12\mathrm{arith}.ext}$ is present, can be determined using the aforementioned criteria [9].

In this scenario, whether the extremum type differs from the non-normalized model remains non-trivial and depends on the functional forms of Y_1 and Y_2 defined by equations (14) or (3), the maximum values $Y_{1\max}$, $Y_{2\max}$, and the ratio between K_1 and K_2.

If $Q_{12\mathrm{arith}.ext}$ cannot be analytically identified, the rational value $Q_{12\mathrm{arith.ratio}}$ and its corresponding argument $x_{12\mathrm{arith.ratio}}$ are determined through direct computational evaluation of $Q_{12\mathrm{arith}}$ via equations (22) and (3) by varying x, whereas the discrepancy between $x_{12\mathrm{arith.ratio}}$ and $x_{11\mathrm{arith.ratio}}$ likewise remains open to investigation.

This stage of the analysis underscores the primary advantage of modeling a composite quality indicator Q via convolution (5)—the weighted arithmetic mean—namely, the linearity of mathematical transformations, which simplifies the analytical identification of the corresponding extremum type (if one exists).

Conversely, the critical limitation of this approach is the explicit dependence of the functional Q extremum on the baseline values $Y_{i.\max}$ during the normalization of Y_i and its transition to y_i. This drawback necessitates the development of a dedicated methodology to establish the relationship between x_{2ext} and x_{1ext}.

In the second stage of the analysis (denoted by the first index "2" at Q), a similar evaluation is performed by representing the composite quality indicator via the weighted harmonic mean according to formula (6). This stage covers both the variant ***without the normalization*** of Y_1 and Y_2 (yielding Q_{12}) and the variant involving their normalization (denoted by the second index "2" at Q_{22}).

Under these conditions, the following analytical expressions are established. **Without the normalization** of Y_1 and Y_2 based on relations (6) and (16), algebraic transformations yield the baseline aggregation function:

$$Q_{21\mathrm{harm}} = \frac{Y_1 \cdot Y_2}{K_1 \cdot Y_2 + K_2 \cdot Y_1}, \tag{26}$$

and similarly (18) the first derivative: and, applying a procedure similar to relation (18), its first derivative is formulated as:

$$\begin{aligned} Q_{21\mathrm{harm}}' = &[(Y_1 \cdot Y_2)' \cdot (K_1 \cdot Y_2 + K_2 \cdot Y_1) \\ &-Y_1 \cdot Y_2 \cdot (K_1 \cdot Y_2' + K_2 \cdot Y_2')] \,/\, (K_1 \cdot Y_2 + K_2 \cdot Y_1)^2, \end{aligned} \tag{27}$$

since, incorporating relation (9), algebraic transformations yield:

$$\frac{Y_1' \cdot Y_2^2}{Y_1^2 \cdot Y_2'} = -\frac{K_2}{K_1}. \tag{28}$$

Equation (28) theoretically defines the argument $x_{21\mathrm{harm}.ext}$ corresponding to the potential extremum $Q_{21\mathrm{harm}.ext}$ (if one exists). Incorporating expressions (14), this parameter is implicitly contained within the components on the left-hand side of equation (28), whereas its right-hand side represents a constant negative ratio.

Utilizing equation (27) as the baseline formulation to derive the second derivative $Q_{21\mathrm{harm}}''$ for characterizing the extremum of $Q_{21\mathrm{harm}}$ leads to the following complex expression:

$$\begin{aligned} Q_{21\mathrm{harm}}'' = &[(K_1 \cdot Y_1' \cdot Y_2^2 + K_2 \cdot Y_1^2 \cdot Y_2')' \cdot \\ &(K_1 \cdot Y_2 + K_2 \cdot Y_1)^2 - 2(K_1 \cdot Y_1' \cdot Y_2^2 + K_2 \cdot Y_1^2 \cdot Y_2') \cdot \\ &(K_1 \cdot Y_2 + K_2 \cdot Y_1)(K_1 \cdot Y_2' + K_2 \cdot Y_1')]/(K_1\, Y_2 + K_2\, Y_1)^4, \end{aligned} \tag{29}$$

which cannot be further simplified, making it impractical for determining the type of extremum. Under these circumstances, the specific type of the $Q_{21\mathrm{harm}}$ extremum can be identified by analyzing the sign change of the first derivative $Q_{21\mathrm{harm}}'$ when varying x in the neighborhood of $x_{21\mathrm{harm}.ext}$. Alternatively, it can be evaluated based on the results of a direct computational analysis of equation (26) by varying x within the range of its feasible practical implementation [9].

When **normalizing** Y_1 and Y_2, the functional form of equation (6) changes as follows:

$$Q_{\mathrm{harm}} = \frac{\sum_{i=1}^{n} K_i}{\sum_{i=1}^{n} \frac{K_i}{y_i}}, \tag{30}$$

and, applying a procedure similar to relation (26) while incorporating expressions (6) and (10), algebraic transformations yield:

$$Q_{22\mathrm{harm}} = \frac{Y_1 \cdot Y_2}{K_1 \cdot Y_{1\max} \cdot Y_2 + K_2 \cdot Y_{2\max} \cdot Y_1}, \tag{31}$$

then, similarly to expression (27), the first derivative is formulated as:

$$\begin{aligned} Q_{22\mathrm{harm}}'' = &[(Y_1\, Y_2)'(K_1\, Y_{1\max}\, Y_2 + K_2\, Y_{2\max}\, Y_1) \\ &-Y_1\, Y_2\, (K_1\, Y_{1\max}\, Y_2' + K_2\, Y_{2\max}\, Y_2')] \,/\, (K_1\, Y_2 + K_2\, Y_1)^2, \end{aligned} \tag{32}$$

since, incorporating relation (9), algebraic transformations yield:

$$\frac{Y_1' \cdot Y_2^2}{Y_1^2 \cdot Y_2'} = -\frac{K_2 \cdot Y_{2\max}}{K_1 \cdot Y_{1\max}}. \tag{33}$$

The parameter value $x_{22\text{harm}.ext}$ corresponding to the potential extremum $Q_{22\text{harm}.ext}$ (if one exists) is implicitly defined by substituting expressions (15) into the left-hand side of equation (33). In this scenario, its right-hand side represents a negative ratio that quantitatively differs from the values calculated via formula (28); thus, in the general case, $x_{22\text{harm}.ext} \neq x_{21\text{harm}.ext}$. This outcome, consistent with the previous stage of the analysis, underscores the explicit dependence of the functional Q extremum on the baseline values $Y_{i\max}$ during the normalization of Y_i and its transition to y_i. Consequently, this limitation necessitates the development of a dedicated methodology to map the relationship between x_{2ext} and x_{1ext}.

To determine the type of extremum, it is theoretically possible to derive the second derivative $Q''_{22\text{harm}}$ using equation (32) as the starting point. However, as observed in the previous computational scenario, this approach yields a structurally complex expression that cannot be further simplified, rendering it impractical for analytical evaluation. Under these circumstances, it is advisable to identify the specific type of the $Q_{22\text{harm}.ext}$ extremum either by analyzing the sign change of the first derivative $Q'_{22\text{harm}}$ when varying x in the neighborhood of $x_{22\text{harm}.ext}$, or, based on the results of a direct computational analysis of equation (31), by varying x within the range of its feasible practical implementation (see above) [9].

In the third stage of the analysis, denoted by the first index "3" at Q, a similar evaluation is performed by representing the composite quality indicator via the geometric weighted average according to formula (7) without the normalization of Y_1 and Y_2. Concurrently, based on relations (7) and (16), this convolution yields:

$$Q_{31\text{geom}} = Y_1^{K_1} \cdot Y_2^{K_2}\,, \tag{34}$$

$$Q'_{31\text{geom}} = K_1\left(Y_1^{K_1-1} \cdot Y_2^{K_2}\right) Y'_1 + K_2\left(Y_1^{K_1} \cdot Y_2^{K_2-1}\right) Y'_2\,, \tag{35}$$

since applying the condition (9) yields:

$$\frac{Y'_1 \cdot Y_2}{Y_1 \cdot Y'_2} = -\frac{K_2}{K_1}, \tag{36}$$

$$\begin{aligned} Q''_{31\text{geom}} = {} & Y_1^{K_1-2} \cdot Y_2^{K_2-2} \cdot [K_1(Y'_1 + Y''_1) \cdot \\ & \left((K_1-1) \cdot Y_2^2 + K_2 \cdot Y_1 \cdot Y_2\right) \cdot K_2(Y'_2 + Y''_2) \cdot \\ & (K_1 \cdot Y_1 \cdot Y_2 + (K_2-1) \cdot Y_1^2)]\,. \end{aligned} \tag{37}$$

The parameter value $x_{31\text{geom}.ext}$ corresponding to the potential extremum $Q_{31\text{geom}}$ (if one exists) is implicitly defined by substituting expressions (14) into the parameters on the left-hand side of equation (36), while the right-hand side represents a constant negative ratio.

Because formula (37) is structurally complex and impractical for analytical application, it is advisable to determine the type of extremum (if present) either by analyzing the sign change of the first derivative $Q'_{31\text{geom}}$ in the neighborhood of $x_{31\text{geom}.ext}$, or via direct computational evaluation of $Q_{31\text{geom}}$ using formula (34) across the feasible range of x.

When normalizing Y_1 and Y_2, the functional form of equation (7) changes as follows:

$$Q_{32\text{geom}} = \left(\prod_{i=1}^{n} y_i^{K_i}\right)^{1/\sum_{i=1}^{n} K_i}, \tag{38}$$

from which, similarly to relation (34) and incorporating formulas (7), (16), and (10), algebraic transformations yield:

$$Q_{32\text{geom}} = \left(\frac{Y_1}{Y_{1\max}}\right)^{K_1} \cdot \left(\frac{Y_2}{Y_{2\max}}\right)^{K_2}. \tag{39}$$

Next, using equation (39), the first derivative $Q'_{32\text{geom}}$ is derived, which can be formulated as:

$$\begin{aligned} Q'_{\text{geom}} = {} & \frac{1}{Y_{1\max}^{K_1-1} \cdot Y_{2\max}^{K_2-1}} \cdot \left[\frac{K_1}{Y_{2\max}} \cdot \left(Y_1^{K_1-1} \cdot Y_2^{K_2}\right) \cdot Y'_1 + \right. \\ & \left. \frac{K_2}{Y_{1\max}} \cdot (Y_1^{K_1} \cdot Y_2^{K_2-1}) \cdot Y'_2\right], \end{aligned} \tag{40}$$

and, after incorporating condition (9) and performing algebraic rearrangements, we obtain:

$$\frac{Y'_1 \cdot Y_2}{Y_1 \cdot Y'_2} = -\frac{K_2}{K_1}. \tag{41}$$

This is identical to expression (36) and indicates independence from the normalization characteristics and specifically for this convolution.

Equation (40) turns out to be structurally similar to formula (35), which leads to:

$$\begin{aligned} Q''_{32\text{geom}} = {} & \frac{Y_1^{K_1-2} \cdot Y_2^{K_2-2}}{Y_{1\max}^{K_1-1} \cdot Y_{2\max}^{K_2-1}} \cdot \left\{\frac{K_1}{Y_{2\max}} \cdot (Y'_1 + Y''_2) \cdot \right. \\ & \left[\left(\frac{K_1}{Y_{2\max}} - 1\right) \cdot Y_2^2 + \frac{K_2}{Y_{1\max}} \cdot Y_1 \cdot Y_2\right] + \\ & \left. \frac{K_2}{Y_{1\max}} (Y'_2 + Y''_2) \left[\frac{K_1}{Y_{2\max}} \cdot Y_1 \cdot Y_2 + \left(\frac{K_2}{Y_{1\max}} - 1\right) Y_1^2\right]\right\}. \end{aligned} \tag{42}$$

Comparison of equations (41) considering formulas (15), (36), and (14) shows that the value x_{32ext} found after normalization of Y_1 and Y_2can be used as a tool for real object management (without normalization). But the difference between formulas (40) and (35), as well as (42) and (37), which is due to the influence of $Y_{i\max}$ on Q when normalizing Y_1 and Y_2, can theoretically invert the identification of the extremum.

In addition, formula (42) turns out to be inconvenient to use. Therefore, in this case, it is advisable to determine the type of extremum (if it exists) by the change of sign of $Q'_{31\text{geom}}$ when varying x near $x_{32\text{geom}.ext}$, or as a result of a direct calculation of the value of $Q_{31\text{geom}}$by formula (39) when varying x in the range of its possible practical implementation.

In the fourth stage of the analysis, denoted by the first index "4" at Q, a similar evaluation is performed by representing the composite quality indicator $Q_{41\text{quad}}$ via the weighted square average according to formula (8) without the normalization of Y_1 and Y_2. In this case, based on relations (8) and (16), this convolution yields:

$$Q_{41\text{quad}} = (K_1 \cdot Y_1^2 + K_2 \cdot Y_2^2)^{0.5}\,, \tag{43}$$

and, after simple algebraic transformations, the first derivative is expressed as:

$$Q'_{41\text{quad}} = \frac{K_1 \cdot Y'_1 \cdot Y_2^2 + K_2 \cdot Y_1^2 \cdot Y'_2}{(K_1 \cdot Y_2 + K_2 \cdot Y_1)^2}, \tag{44}$$

from which, after incorporating condition (9), we deduce:

$$\frac{Y'_1 \cdot Y_1}{Y_2 \cdot Y'_2} = -\frac{K_2}{K_1}. \tag{45}$$

The parameter value $x_{41\text{quad}.ext}$ corresponding to the potential extremum $Q_{41\text{quad}.ext}$ (if one exists) is implicitly defined by substituting expressions (14) into the parameters on the left-hand side of equation (45), while the right-hand side represents a constant negative ratio.

The general representation of the second derivative:

$$\begin{aligned} Q''_{41\text{quad}} = {} & [(K_1\, Y'_1\, Y_2^2 + K_2\, Y_1^2\, Y'_2)' \cdot (K_1\, Y_2 + K_2\, Y_1)^2 \cdot \\ & -(K_1\, Y_2^2\, Y'_1 + K_2\, Y_1^2\, Y'_2) \cdot 2\ \cdot (K_1\, Y_2 + K_2\, Y_1) \cdot \\ & (K_1\, Y'_2 + K_2\, Y'_1)]/(K_1\, Y_2 + K_2\, Y_1)^4 \end{aligned} \tag{46}$$

does not allow simplification, turning out to be cumbersome and impractical for determining the type of extremum $Q_{41\text{quad}.ext}$. Therefore, in this case, it is advisable to determine the type of extremum (if it exists) by analyzing the sign change of the first derivative $Q'_{41\text{quad}}$ in the neighborhood of $x_{41\text{quad}.ext}$ or via direct computational evaluation of $Q_{41\text{quad}}$ using formula (43) across the feasible range of x.

When normalizing Y_1 and Y_2, the functional form of equation (8) changes as follows:

$$Q_{42\text{quad}} = \sqrt{\frac{\sum_{i=1}^{n} K_i \cdot y_i^2}{\sum_{i=1}^{n} K_i}}. \tag{47}$$

Based on this, similarly, considering relations (8), (16), and (10), we obtain:

$$Q_{42\text{quad}} = \left(\frac{Y_1}{Y_{1\max}}\right)^{K_1} \cdot \left(\frac{Y_2}{Y_{2\max}}\right)^{K_2}, \tag{48}$$

$$Q'_{42\text{quad}} = \frac{1}{Y_{1\max}\, Y_{2\max}} \cdot \frac{K_1\, Y_{2\max}^2\, Y_1\, Y'_1 + K_2\, Y_{2\max}^2\, Y_2\, Y'_2}{(K_1\, Y_{2\max}^2\, Y_1^2 + K_2\, Y_{1\max}^2\, Y_2^2)^{0.5}}, \tag{49}$$

and, incorporating condition (9), we obtain:

$$\frac{Y'_1 \cdot Y_1}{Y_2 \cdot Y'_2} = -\frac{K_2 \cdot Y_{1\max}^2}{K_1 \cdot Y_{2\max}^2}. \tag{50}$$

The parameter value $x_{42\text{quad}.ext}$ corresponding to the potential extremum $Q_{42\text{quad}.ext}$ (if one exists) is implicitly defined by substituting expressions (15) into the left-hand side of equation (50). Concurrently, the right-hand side yields a negative ratio that quantitatively differs from the value obtained via formula (45); thus, in the general case, $x_{42\text{quad}.ext} \neq x_{41\text{quad}.ext}$. Consistent with the previous stages of the analysis, this outcome demonstrates the explicit dependence of the functional extremum Q on the baseline values $Y_{i\max}$ during the normalization of Y_i and its transition to y_i, thereby necessitating the development of a dedicated methodology to establish the relationship between x_{2ext} with x_{1ext}.

The second derivative $Q''_{42\text{quad}}$ turns out to be even more structurally complex and impractical for analytical evaluation than $Q''_{41\text{quad}}$ derived via formula (46). Therefore, it is advisable to determine the type of extremum (if one exists) either by analyzing the sign change of the first derivative $Q'_{42\text{quad}}$ in the neighborhood of $x_{42\text{quad}.ext}$, or through a direct computational analysis of $Q_{42\text{quad}}$ via formula (48) by varying x within its feasible practical implementation range.

4. DISCUSSION OF RESULTS

Overall, the presented findings indicate that the normalization of individual quality indicators Y_1 and Y_2 affects not only the absolute value of the composite quality indicator Y_1 and Y_2—when modeled using arithmetic, harmonic, and quadratic weighted means—but also the magnitude and type (maximum or minimum) of the corresponding extremum Q_{ext}.

This influence also manifests as a shift in the optimal value of the practical quality management tool (x_{ext},), which determines the corresponding extremum Q_{ext}.

Concurrently, this study reveals for the first time that the optimal management tool value (x_{ext}) remains independent of the normalization of Y_1 and Y_2 when the composite quality indicator Q is modelled using a weighted geometric mean convolution. This finding suggests that the geometric mean is highly suitable for quality modelling, as it allows the optimal tool parameters x_{ext} (or the corresponding rational value x_{ratio}) determined under normalized conditions to be directly applied to real-world object management without requiring inverse transformation. However, close attention must still be paid to correctly determining the specific type of the Q_{ext} extremum.

Consequently, there is a clear need to develop dedicated mathematical methodologies to establish the exact relationship between x_{2ext} and x_{1ext} (or X_{2ext} and X_{1ext}). This will ensure precise operational management of the object based on the results of quality modeling using normalized parameters. Furthermore, the specific functional dependencies of Y_1 and Y_2 on the argument x are expected to significantly affect the resulting parameters of the composite quality indicator.

5. CONCLUSIONS

1. The rational management of technical, technological, or organizational systems requires determining the specific conditions and optimal levels of quality management tools. These tools shape individual quality indicators with their respective weighting coefficients, which, in turn, form a composite quality indicator for the object. This relationship between individual indicators and management tools exists objectively, and its specific analytical representation must be defined by an expert.

2. Incorporating heterogeneous individual quality indicators and management tools—which possess disparate natures, dimensions, and variation ranges—into a quality model requires reducing them to a comparable form. This is achieved by normalizing them relative to baseline values, accounting for the directional impact (increase or decrease) of individual indicators on the composite quality metric, as well as the relationships between these indicators and the management tools.

3. The normalization of individual quality indicators (and potentially management tools) yields distinct qualimetric models of the same object. The quantitative values of the composite quality indicators in these models are generally incomparable within a single scale of relations.

4. In the absence or extreme complexity of an analytical dependence of the composite quality indicator on its parameters,

the quality model is typically constructed using mathematical convolutions, such as weighted means (arithmetic, harmonic, geometric, or quadratic). This approach requires identifying the discrepancies between the normalized and non-normalized model parameters and developing dedicated methods for back-transformation. This ensures rational, real-world management aimed at maximizing the composite quality indicator within the permissible range of management tool levels.

5. For analytical purposes, this study adopted a simplified model representing the dependence of the composite quality indicator on two individual (both non-normalized and normalized) quality indicators that depend on a single common management tool and exert opposing effects on the composite metric. The composite indicator was evaluated using four distinct weighted mean convolutions: arithmetic, harmonic, geometric, and quadratic.

6. The analytical results reveal for the first time the specific conditions under which the normalization of individual quality indicators influences the presence and type of the composite indicator's extremum, as well as the optimal management tool values determined by this extremum.

7. The analysis reveals for the first time that the optimal level of the object quality management tool remains independent of individual indicator normalization when modelling the composite quality indicator via a weighted geometric mean convolution. Consequently, the geometric mean is highly recommended for quality modelling, as it allows the optimal tool parameters determined under normalized conditions to be directly applied to real-world object management (i.e., without normalization). However, careful attention must still be paid to correctly identifying the type of extremum.

8. The findings obtained can enhance the efficiency of quality management for multi-parameter real-world systems by identifying the conditions required to maximize the composite quality indicator depending on the selected convolution type, functional dependencies, and normalization procedures of individual quality indicators.

9. Future research will focus on analysing specific functional dependencies of individual quality indicators on management tools, and on developing methods to relate the rational levels of quality management tools in normalized and non-normalized models. The approach can also be extended to systems with more than two quality indicators and management tools.

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