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Statistical monitoring of atomic clocks: Applying Hotelling's *T*-squared statistic and exponentially weighted moving average to caesium standards

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ABSTRACT
Ensuring the reliability of commercial caesium standards is essential, as anomalies in these atomic clocks can compromise critical systems that rely on precise timekeeping. Instabilities in caesium standards may lead to failures in satellite navigation, military communications, network synchronisation, and atomic time scales maintained by laboratories worldwide. Given their role as the foundation of atomic time, due to their high stability and precision, continuous monitoring of caesium standards is necessary to preserve signal integrity. This study applies statistical process control techniques to detect early signs of anomalous behaviour in caesium standards used in the Brazilian atomic time scale. Specifically, univariate and multivariate control charts are employed, including Hotelling's T^2 statistical test to identify deviations and determine their onset before a significant instability occurs. Additionally, an Exponentially Weighted Moving Average (EWMA) plot is used to monitor individual parameters within the multivariate framework. The results confirm abnormal behaviour in one of the standards and precisely pinpoint the moment the anomaly begins, demonstrating the effectiveness of the combined use of Hotelling's T^2 and EWMA methods.

Section: RESEARCH PAPER

Keywords: caesium standard; EWMA; Hotelling's T^2; control charts; time scale; measure

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1. INTRODUCTION

Primary laboratories use atomic clocks to generate an independent atomic time scale [1]. A time scale is a system which unequivocally assigns a temporal coordinate, called a date, to any event [2].

The precision of the caesium standards allowed the development of the current time scale, simplifying the understanding of the relativistic effect [3].

Coordinated Universal Time (UTC), which serves as the international reference time scale, is computed by the Bureau International des Poids et Mesures (BIPM) using data from various atomic clocks maintained in more than 80 institutions [4]. It is essential that these watches are highly stable, accurate and reliable. The commercialisation of caesium standards allowed countries to create their own time scales, which serve as a stable reference for calibration and construction of atomic time scales [5]

While it is possible to create a time scale with a single clock, it is not recommended due to the risks associated with failures [6]. It is essential to use several clocks to compose a time scale, the data of which are taken by an algorithm that computes a weighted average. The result of this calculation is used to correct the accuracy of the signal generated by the scale [7].

In Brazil, the Brazilian Legal Hour Services Division (DISHO) of the National Observatory (ON) is responsible for the atomic time scale. The Brazilian Atomic Time Scale, ETAB1(ONRJ), used by ON, employs commercial caesium beam, caesium optical atomic, and hydrogen maser standards for its realisation. Since 1989, UTC(ONRJ) has been compared to UTC(BIPM) using a GPS receiver, and the results are sent to

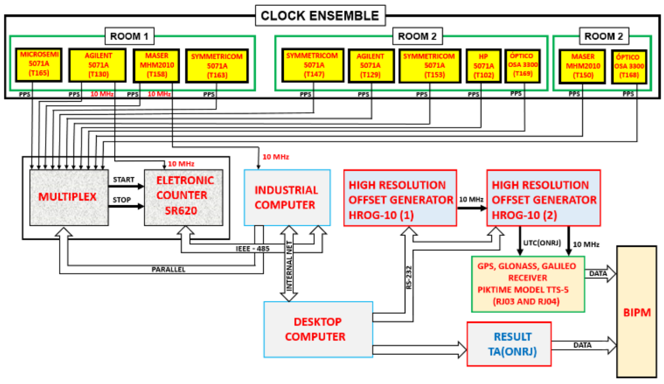

Figure 1. Block diagram (updated) of the Brazilian atomic time scale system of the National Observatory - ON.

BIPM [1]. Figure 1 shows the block diagram of the ON's Brazilian Atomic Time Scale.

Techniques are applied to detect anomalies in caesium patterns to keep a country's time scale in full working order. These devices are widely used in defining UTC and are maintained by experts in atomic time scales, ensuring the accuracy and stability of the time reference.

1.1. General objectives

The stability of caesium standards is essential for the accuracy and reliability of time measurement devices. In the long term, the commercial caesium standard has a stability of approximately 4×10^{-12} [8]. If this stability is not guaranteed, time measurements can become inaccurate, affecting areas such as telecommunications and navigation, which depend on accurate synchronisation. The Allan deviation is a widely used statistic to estimate frequency stability, allowing the identification of changes in the frequency of an oscillator over time [9]. Significant variations in the stability of caesium standards can result in anomalous behaviour in the phase difference between the Pulse-per-Second (PPS) signals of the standards, as shown in Figure 2. These significant variations in the red part of the graph are due to anomalies in the standard, which are indicated in the parameters.

The objective of this study is to detect anomalies in the caesium standard parameters using data science techniques, such as Principal Component Analysis (PCA), Exponential Weighted Moving Average (EWMA), and Hotelling's T^2, to anticipate corrective actions and preserve the stability of the output signals, ensuring the integrity of the atomic time scale.

1.2. Specific objectives

- Use multivariate PCA techniques to reduce the study variables, preserving information;
- Create univariate EWMA and multivariate Hotelling's T^2 control charts to verify statistical control of standard parameters;
- Identify the onset of anomalies through EWMA or Hotelling's T^2 control charts;
- Estimate the time between the detection of the anomaly and the compromise of the stability of the pattern;
- Observe the behaviour of the Allan deviation before and after the anomaly detection to verify the stability of the frequency.

2. LITERATURE REVIEW

2.1. Related Work

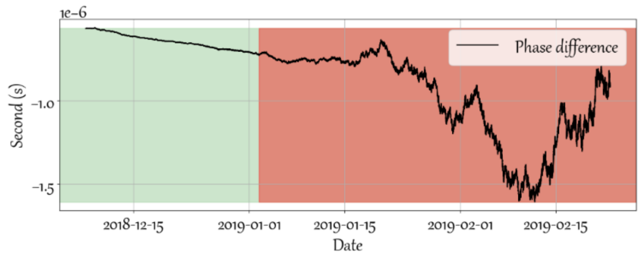

Figure 2. Anomalous behaviour of the phase difference between the PPS signal of the reference standard and the PPS signal of the standard under analysis. We can verify the significant variation due to the change in stability.

Hewlett-Packard implemented technical improvements in its caesium standards, allowing remote monitoring of its parameters for analysis [10]. This improvement supported a study carried out at the United States Naval Observatory (USNO), in which differences in the values of certain parameters between different standards were found, however, these differences do not indicate anomalies. Chadsey and Kubik [10] identified seven parameters (E-multiplier, Ion Pump, Osc. Control, Signal Gain, Thermometer, RF amplitude 1, and RF amplitude 2) as important to predict the performance and useful life of the patterns.

2.2. Principal Component Analysis (PCA)

PCA is a statistical technique that reduces the dimensionality of data, capturing its variability in uncorrelated principal components, supporting the interpretation and visualisation of results [2]. It is widely used in many areas to explore patterns and make informed decisions. It is a powerful tool for exploratory data analysis and decision making in many areas, from scientific studies to practical applications in business and engineering.

2.3. Hotelling's T^2 multivariate control chart

In 1947, Hotelling pioneered control chart techniques to track multiple quality characteristics simultaneously [11]. Hotelling's T^2 control chart is a technique widely used in industry to monitor the variance of a multivariate population by detecting anomalous data points based on measures of variance and covariance. However, it's important to carry out additional testing and analysis to confirm issues and determine their source, according to Montgomery [12].

The use of control charts involves the application of phases I and II to establish control limits and monitor the process. In phase I, data is analysed to verify process control, while in phase II, they are used to monitor the process, comparing sample statistics to control limits. The Hotelling statistic is a measure which combines the estimated mean vector $\bar{\bar{X}}$ and the estimated covariance matrix (S) of the $\bar{\bar{X}}$ vector under control of the process mean vector [13] according to equation (1):

$$T^2 = n\left(\bar{X} - \bar{\bar{X}}\right)^t S^{-1}\left(\bar{X} - \bar{\bar{X}}\right). \tag{1}$$

For phase I, calculations of UCL and LCL control limits are given by

$$UCL = \frac{p\,(m-1)\,(n-1)}{m\,n - m - p + 1}\,F_{\alpha,\mathrm{p,m\,n-m-p+1}} \tag{2}$$

$$LCL = 0\,, \tag{3}$$

where $F_{\alpha,\mathrm{p,m\,n-m-p+1}}$ is the percentile of the F-distribution with p and $m\,n - n - p + 1$ degrees of freedom.

For phase II, the calculations of the new UCL and LCL limits are established only to monitor future observations [14], and the control limits are given by

$$UCL = \frac{p\,(m+1)\,(n-1)}{m\,n - m - p + 1}\,F_{\alpha,p,m\,n-m-p+1} \tag{4}$$

$$LCL = 0\,. \tag{5}$$

2.4. Exponentially Weighted Moving Average (EWMA) control chart

As defined by Montgomery [12], the EWMA control chart is used to calculate the exponentially weighted moving average when we specifically want to detect small changes. It is typically used with individual observations. This approach allows detecting small changes in the process mean. To calculate the exponentially weighted moving average, equation (6), and the control limits, equations (7) and (8), we use the standard equations recommended in the literature [12]–[15]:

$$z_i = \lambda\, x_i + (1-\lambda)\, z_{i-1} \tag{6}$$

$$UCL = \mu_0 + L\,\sigma \sqrt{\frac{\lambda}{2-\lambda}\,[1-(1-\lambda)^{2\,i}]} \tag{7}$$

Centre line = μ_0

$$UCL = \mu_0 - L\,\sigma \sqrt{\frac{\lambda}{2-\lambda}\,[1-(1-\lambda)^{2\,i}]}\,, \tag{8}$$

where L is the width of the control limits ("Limits of L-sigma") and σ is the standard deviation when the process is in control.

Hotelling's T^2 control chart and the EWMA control chart are tools used to monitor statistical processes. These tools were used in phase I of this research to monitor the caesium standard parameters. These tools were useful to detect anomalies and to verify that the processes are within the established limits.

3. MATERIAL AND METHOD

30 files (58460.txt - 12/08/2018 to 58490.txt - 01/07/2019) containing records collected from minute to minute were used. 22 files (58460.txt - 12/08/2018 to 58481.txt - 12/29/2018) were used in phase I and 9 files (58482.txt - 12/30/2018 to 58490.txt - 01/07/ 2019) in phase II, respectively. A Python algorithm was developed to concatenate the files used in phase I and phase II and transform them into dataframes. Missing data were replaced using interpolation techniques. The two dataframes contained the standard deviation of the parameters. Each standard deviation value was calculated according to equation (9):

$$s_j = \sqrt{\frac{\sum_{i=j}^{(n-1)+j}(x_i - \bar{x})^2}{n-1}}\,, \tag{9}$$

where n is the window size for calculation, in this case, $n = 60$. The j is the displacement one step ahead ($j = 0, 1, 2, 3, \ldots, m$), where m is in this case the total value of register in the dataframe.

The first dataframe was used to build the multivariate control chart with Hotelling's T^2 and to build the univariate EWMA control charts for the four parameters of the caesium standard, considered phase I. To determine this phase, past data in which the caesium standard presented parameters "under control" were used. In Hotelling's T^2 plot, the dataframe dataset was submitted to PCA to decompose the matrix of variances and covariances of the data into principal components. The second dataframe was used to build the future data graphs in phase II.

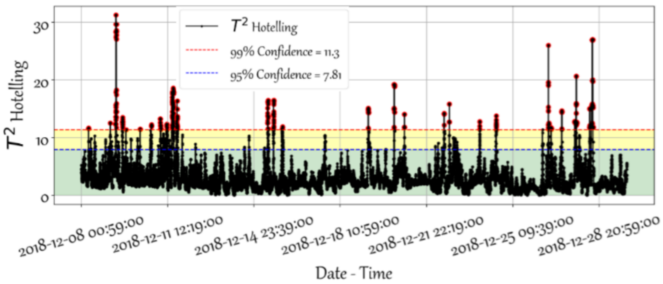

Figure 3. Hotelling's T^2 control chart for phase I, obtaining 95% and 99% confidence limits.

3.1. Selection of caesium standard parameters

The parameters selected for the analysis were Osc. Control, RF-Amplitude 1, RF-Amplitude 2, and E-multiplier. These parameters were chosen based on their temporal evaluation, as it was noticed that they show significant variations in terms of values at practically the same moment. These variations were not observed in the other parameters or occurred at much later times.

3.2. Building the control charts (phase I)

After the analysis, data were split into two dataframes and missing values were removed. In the first dataframe, multivariate control charts have been built using Hotelling's T^2 and univariate EWMA control charts. The PCA applied to this dataframe showed that three principal components explain 91.4 % of the variability.

3.2.1. The T^2 Hotelling chart

Figure 3 shows the dataframe with three principal components and with 95.0 % (T^2 = 7.81) and 99.0 % confidence limits (T^2 = 11.3).

3.2.2. The EWMA charts

For each parameter of the caesium standard, an EWMA control chart was built (Figure 4). For all caesium standard parameters, the following values were assigned to the function parameters: *Lambda* = 0.05, L = 2.7, and *Sigma* = 1.

Table 1 shows the calculated control limits when parameter data were submitted to the EWMA control charts.

The Hotelling T^2 and the EWMA control charts are tools to monitor statistical processes and were used in phase I to monitor the parameters of the caesium standard when it behaved normally.

The 95 % and 99 % confidence limits of the Hotelling T^2 control chart and the UCL, CL, and LCL control limits of the EWMA control charts obtained in phase I will be used in phase II to observe the statistical control of the parameters of the caesium standard for future data.

Hotelling's T^2 control chart was used to monitor the variability of data from the PCA. The EWMA control chart was used to monitor the individual stability of each parameter.

These tools were useful to detect anomalies in the process and to verify if the processes are within the established limits.

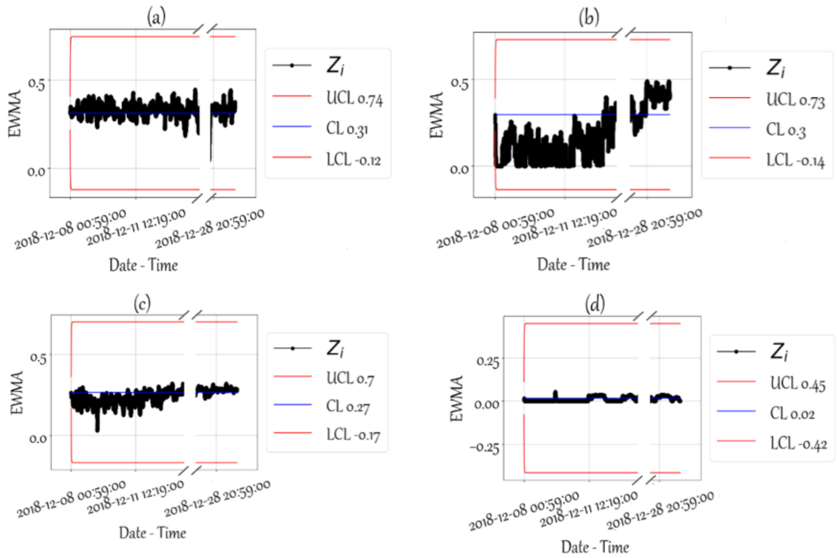

Figure 4. Control charts for parameters Osc Control (a), RF-amplitude 1 (b), RF-amplitude 2 (c) and E-multiplier (d) using Exponential Weighted Moving Average (EWMA) with its control limits.

4. RESULTS AND DISCUSSION

4.1. Anomaly detection – Hotelling's T^2 control chart (phase II)

In phase II of the control chart analysis, a second dataframe consisting of 12,960 records from 9 files was used. These files contained future data ranging from 30/12/2018 to 07/01/2019. The analysis also included the standard deviation of the data.

During the analysis, an anomaly was detected at 11:17 AM on January 1, 2019, as shown in Figure 5. Following this point, there was a noticeable escalation in process instability, with a significant increase in the values of each subsequent data point.

To support data visualisation, an anomaly ellipse was built using the second and third principal components, as depicted in Figure 6.

To investigate which specific parameter contributed to the detected anomaly, the parameters Osc. control, RF-amplitude 1, RF-amplitude 2, and E-multiplier were analysed using the EWMA control charts (Figure 7).

The EWMA charts highlighted the moments of the first anomaly detections for each parameter: the E-multiplier had its first anomaly at 1:32 PM on January 1, 2019; RF-amplitude 1 at 8:36 AM on January 2, 2019; Osc. control at 10:04 AM on the same day; and RF-amplitude 2 at 11:39 AM on January 2, 2019.

These results highlight that the E-multiplier was the first parameter showing significant deviation, preceding the other parameters and potentially affecting the multivariate anomaly flagged by Hotelling's T^2.

When analysing the phase difference graph between PPS signals, the control charts were found to detect anomalies before major variations occurred in the phase difference itself – a finding illustrated in Figure 8.

Table 1. Lower control limits (LCL), centre line (CL) and upper control limits (UCL) using EWMA for each parameter of the caesium standard.

	LCL	CCL	UCL
Osc. Control (a)	-0.12	0.31	0.74
RF-amplitude 1 (b)	-0.14	0.30	0.73
RF-amplitude 2 (c)	-0.17	0.27	0.70
E-multiplier (d)	-0.42	0.02	0.45

Additionally, the Allan deviation values were examined before and after the detected anomaly. The Allan deviation remained below $\sigma = 2 \cdot 10^{-11}$ in the days surrounding the detection. However, starting from the 18th day, specifically on January 19, 2019, a significant increase in Allan deviation was observed, as shown in Figure 9.

4.2. Discussion

The results confirm the effectiveness of Hotelling's T^2 and the EWMA control charts in detecting anomalies in caesium frequency standards before significant performance degradation occurs.

A particularly relevant finding is that Hotelling's T^2 chart identified the anomaly on January 1, while the Allan deviation – a traditional frequency stability metric – only reflected a major deviation 18 days later. This demonstrates the predictive power of the proposed monitoring approach, allowing corrective actions to be considered well in advance.

The use of EWMA charts allowed the identification of the specific parameter responsible for the initial anomaly, with the E-multiplier being the earliest indicator. This highlights the complementary strength of combining univariate and multivariate techniques in monitoring atomic standards.

The comparison between the control charts and the phase difference graph reinforces the sensitivity of statistical process control tools for early anomaly detection. Even before visual evidence appears in traditional phase comparisons or Allan deviation plots, the statistical tools have already identified deviations beyond the control thresholds.

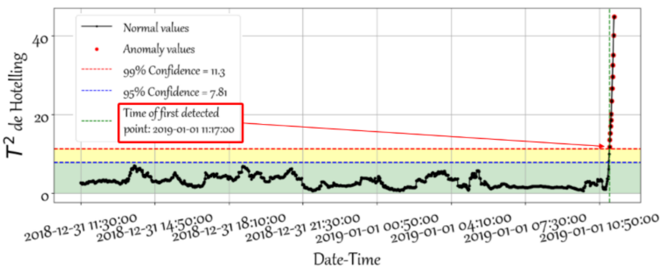

Figure 5. Hotelling's T^2 Control Chart identifying anomaly data. The first abnormal data point was detected at 11:17 AM on January 1, 2019.

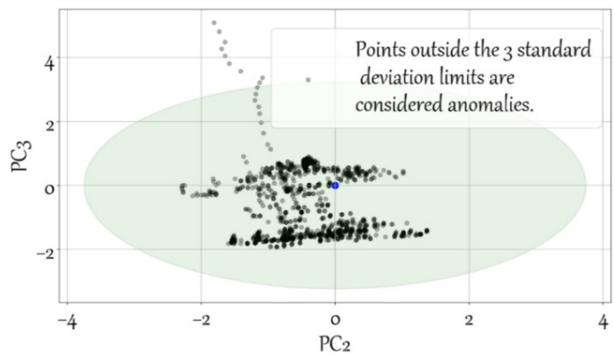

Figure 6. The ellipse plot shows that some data points are outside the three standard deviation limits. These data points are considered anomalies.

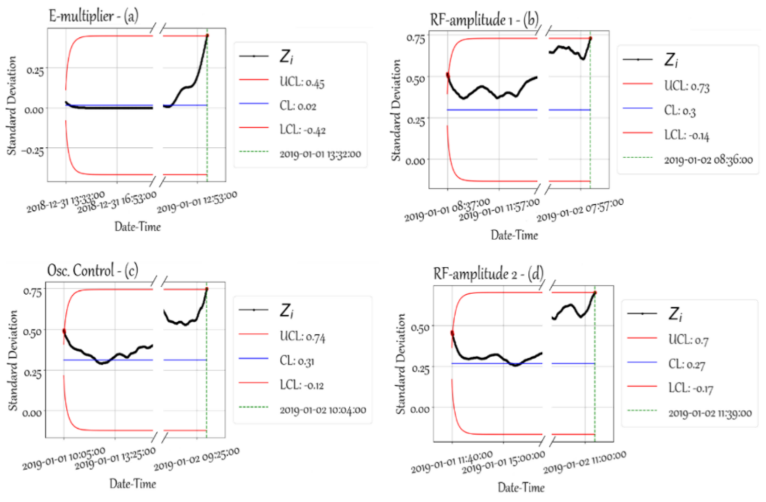

Figure 7. Moment when the detections occurred in the parameters E-multiplier (a), RF-amplitude 1 (b), Osc. Control (c) and RF-amplitude 2 (d).

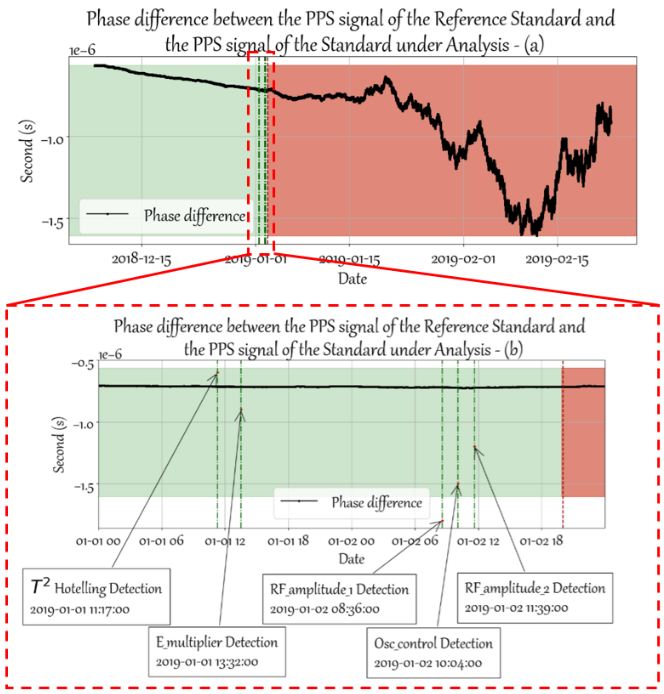

Figure 8. (a) - Graph of the phase difference between the PPS signal of the reference standard and the PPS signal of the standard under analysis. (b) - Time instances in which each parameter exhibited anomalies in its data. Verifying the stability of the caesium standard after detecting the Hotelling's T^2 control chart.

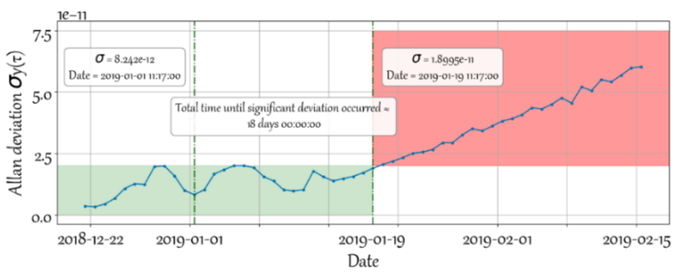

Figure 9. Frequency stability graph displaying only the Allan deviation values for τ = 60 s. In the green region, the Allan deviation is under control. In the red region, the Allan deviation is out of control.

These insights validate the practical application of statistical control charts in time and frequency metrology, and support the idea that such methods can serve as an early warning system for instabilities in atomic standards.

Although this study uses classical statistical techniques, such as Hotelling's T^2 and the EWMA, machine learning-based approaches, such as neural networks, autoencoders, and others, have proved promising for high-accuracy anomaly detection in time series. Future research could explore the integration of these approaches to compare performance and sensitivity, especially in contexts with multiple correlated variables and nonlinear variations.

5. CONCLUSION

This work highlighted the utility of multivariate control charts (Hotelling's T^2) and univariate control charts (EWMA) for anomaly detection in atomic caesium standards. The control charts were able to identify anomalies before the frequency stability of the standard was compromised, emphasising the importance of precise and continuous monitoring.

The T^2 Hotelling chart detected the first anomaly at 11:17 AM on January 1, 2019. The EWMA charts for individual parameters also identified anomalies, with the E-multiplier, RF-amplitude 1, Oscillator Control, and RF-amplitude 2, each showing deviations at different times between January 1 and January 2.

The use of PCA to reduce dimensionality prior to applying the T^2 Hotelling chart proved more effective than monitoring each parameter in isolation. Nonetheless, the individual EWMA charts enhanced detection sensitivity and helped identify which parameter exhibited the earliest abnormal behaviour.

Importantly, the caesium standard remained apparently stable for 18 days after the initial anomaly was detected by the control charts, with significant changes in Allan deviation only appearing on January 19. This time window demonstrates the method's potential for early intervention to prevent disruptions in time scale continuity.

Future work may explore the application of this methodology to more advanced timekeeping systems, such as optical clocks, which demand even higher precision and real-time monitoring. Additionally, integrating machine learning techniques – such as anomaly detection with neural networks or unsupervised clustering – could further improve the sensitivity and adaptability of the monitoring process. These approaches may offer more dynamic responses in complex, non-linear systems and could be particularly useful in scenarios with large volumes of high-resolution data.

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