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Probability theory as a logic for modelling the measurement process

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ABSTRACT
The problem of the nature of probability has been drawn to the attention of the measurement community in the comparison between the frequentist and the Bayesian views, in the expression and the evaluation of measurement uncertainty. In this regard, it is here suggested that probability can be interpreted as a logic for developing models of measurement capable of accounting for uncertainty. This contributes to regard measurement theory as an autonomous discipline, rather than a mere application field for statistics. Following a previous work in this line of research, where only measurement representations, through the various kinds of scales, were considered, here the modelling of the measurement process is discussed and the validity of the approach is confirmed, which suggests that the vision of probability as a logic could be adopted for the entire measurement theory. With this approach, a deterministic model can be turned into probabilistic by simply shifting from a deterministic to a probabilistic semantic.

Section: RESEARCH PAPER

Keywords: measurement theory, epistemology, philosophy of probability, measurement modelling, probabilistic models

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1. INTRODUCTION

The problem of the nature of probability has become a topic in measurement science for over twenty years, as a part of the debate on the expression and evaluation of measurement uncertainty, significantly raised by the publication of the Guide to the expression of uncertainty in measurement (GUM) [1] and of its long lasting and still ongoing revision process [2]. In the debate, the opposition between the Bayesian and the frequentist schools of thoughts in statistics soon emerged, which involves the consideration of the nature of probability. In this regard, some authors pursue an explicit adoption of a Bayesian paradigm for the overall context of uncertainty evaluation [3], [4], others instead suggest maintaining a more open attitude [5], [6], even when expressing a preference for the Bayesian view [7], or to include the frequentist approach [8], when appropriate.

Here the focus is put on measurement modelling and probability is regarded as a logical and mathematical tool for developing such models, in such a way as to account for uncertainty. Alternative choices could be done, for example those based on the evidence theory [9], [10], but here only probability is discussed and investigated. Measurement modelling has been recently the subject of investigation, not only in respect to practical issues [11], but also to theoretical and foundational aspects [12], [13]. Yet the "nature" of probability in such modelling seems to have not been discussed explicitly, which is instead the goal of this communication. Basically, it is here suggested that probability can be regarded as an appropriate logic for developing models of measurement when uncertainty must be accounted for.

Therefore, in Section 2 deterministic measurement modelling will be firstly addressed. Then, in Section 3, the logical approach to probability here proposed will be presented. Its application to probabilistic measurement modelling will be addressed in Section 4 and conclusions will be drawn in Section 5.

2. DETERMINISTIC MEASUREMENT MODELLING

2.1. Generic modelling issues

It is here suggested that probability can be understood as a logic for developing measurement models. The notion of model thus needs reviewing. To establish some terminology, let us consider a system as a set of entities with relations among them [14]. A model can be thus understood as an abstract system, capable of describing a class of real systems. For example, if we consider the height of the inhabitants of a generic town, the model, M, can be expressed by a function, $h: U \rightarrow X$, that associate to each inhabitant his/her height, on a proper height scale. For maximum simplicity, in the following illustrative examples height will be considered as a purely ordinal property, and X the set of the numbers expressing height on an ordinal scale. Therefore, the model can be synthetically expressed by the triple:

$$M = (U, X, h). \tag{1}$$

Let us now introduce the distinction between deterministic and probabilistic models. A typical statement related to model M is:

$$h(u) = x\ , \tag{2}$$

with $u \in U$ and $x \in X$. Yet the truth of this statement is undefined, till a *specific* town, T, is considered. With reference to T, instead, if A denotes the set of its inhabitants, X_A the set of their height values and h_A the corresponding height function, the model is now specialised to T, that is:

$$M(T) = (A, X_A, h_A). \tag{3}$$

Suppose for example that in T there are just 3 inhabitants, $A = \{a, b, c\}$, that $X = \{1,2\}$, and $h_A = \{(a, 2), (b, 1), (c, 1)\}$, then

$$(A, X_A, h_A) = (\{a, b, c\}, \{1,2\}, \{(a, 2), (b, 1), (c, 1)\}). \tag{4}$$

The structure in Equation (4) provides a sematic, that is a criterion of truth for the deterministic model M, since it allows us to ascertain the truth of any statement involved in the model. For example, $h(a) = 2$ is true, whilst $h(b) = 2$ is false. The general truth criterion is thus, for town $T, u \in A, x \in X_A$:

$$h(u) = x \leftrightarrow (u, x) \in h_A. \tag{5}$$

Let us call T an instance of the model M: then a model is deterministic if for any instance of the model all the statements concerning the model are either true or false. Conversely, we will call probabilistic a model where for at least one of its instances there is at least one statement concerning the model for which its state of truth cannot be ascertain, but only a probability can be assigned to it. The transition from a deterministic to a probabilistic description will be discussed in Section 3.

2.2. Modelling the measurand

Measurement modelling concerns both the measurand and the measurement process. The modelling of the measurand aims at ensuring that the property of interest can be measured and it is thus closely related to the measurability issue [15]. At a foundational level, this implies assuming that the quantity[1] under consideration can be measured on an appropriate measurement scale, i.e., that it possesses the required empirical properties. For example, (empirical) order is required for an ordinal scale, whilst order and difference are needed in the case of an interval scale. At a more operational level, modelling the measurand may account for the interactions it has with the environment and with the measuring system, to ensure that they do not hinder measurement, to compensate them, if possible, and to account for them in the uncertainty budget.

Here only the first aspect, that is the possess of proper empirical properties, is briefly discussed. This is typically the scope of the so-called representational theory and it can be summarised by one or more representation theorems. For example, taking again the case of height of persons, still considering it as an ordinal property, an operation of empirical comparison needs considering, that allows us to determine, for any pairs of persons, u and v, whether u is taller than v, $u \succ_h v$, or v is taller than u, $v \succ_h u$, or they are equally tall, $u \sim_h v$. One such operation, provided that is transitive, ensures that the function “height of persons”, introduced in the previous section, exists. The corresponding model is now:

$$M' = (U, X, \succcurlyeq, m)\ , \tag{6}$$

where h, height, has been replaced by the more general symbol m, “measure”, and the subscript h has been dropped accordingly. Then, the corresponding representation theorem reads:

$$u \succcurlyeq v \leftrightarrow m(u) \geq m(v)\ . \tag{7}$$

Yet, although the existence of the function h is mathematically ensured by the properties of the empirical relation $\succcurlyeq_h$, its actual experimental determination requires a measurement process, which is to be modelled now.

2.3. Modelling the measurement process

For modelling the measurement process, the approach proposed in Reference [16] is here followed and only very briefly recalled. It is suggested that measurement can be parsed in two phases, called *observation* and *restitution*. In the observation phase the “object”[2] carrying the property to be measured interacts with the measurement system, in such a way that an observable output, called instrument indication, is produced, based on which a measurement value can be assigned to the measurand. The successive phase, where the result is produced, based on the instrument indication and accounting for calibration results (calibration curve), is here called restitution. This approach is essentially in agreement with others recently proposed in the literature [17]-[20]. For example, in the case of persons’ height, the measuring device may consist of a platform, on which the subject to be measured must stand erect, and of an ultrasonic sensor, placed at a fixed height over the head of the subject. The instrument generates a signal whose intensity, y, is proportional to the distance of the sensor from the top of the head of the subject, which constitutes the instrument indication.

Let us call φ the function that describes this phase: thus, if a is the object to be measured,

$$y = \varphi(a). \tag{8}$$

Calibration requires the pre-constitution of a reference (measurement) scale, $R = \{(s_1, x_1), (s_2, x_2), \dots, (s_n, x_n)\}$, which includes a set of standards, $S = \{s_1, s_2, \dots, s_n\}$, and their corresponding measurement values, $X = \{x_1, x_2, \dots, x_n\}$. Calibration can be done by inputting such standards to the measuring system and recording their corresponding indications, thus forming the function $\varphi_s = \{(s_1, y_1), (s_2, y_2), \dots, (s_n, y_n)\}$, which is a subset of φ, defined above. Based on this information, it is possible to obtain a calibration function, $f =$

[1] “Quantity” here stands for “measurable property”.

[2] Note that here the term “object” has to be understood as “the carrier of the property to be measured” irrespectively of it being a concrete object, like a workpiece, or an event, such as a sound or a shock, or even a person, in the case of psychometrics [13].

$\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, that establishes a correspondence between the value of each standard and the corresponding output of the measuring device. Calibration allows us to perform measurement, since, once the instrument indication has been obtained, it is possible to assign to the measurand, in the restitution phase, the value of the standard that would have produced the same output, that is

$$\hat{x} = f^{-1}(y) \triangleq g(y). \tag{9}$$

Lastly, we obtain a description of the overall measurement process, by combining observation and restitution:

$$\hat{x} = \gamma(a) = g\big(\varphi(a)\big) = f^{-1}\big(\varphi(a)\big). \tag{10}$$

This equation constitutes a basic deterministic model of the measurement process. In [20], a more detailed model was presented, where the generation of instrument indication was more deeply investigated. Yet the structure of that model is compatible with the one just recalled, that will be used in the following, for the sake of simplicity.

Let us now show how this model can be turned into probabilistic by just shifting from a deterministic to a probabilistic semantic, which is the main goal of this communication. Prior to doing so, the present approach to considering probability theory as a logic must be presented, with a special focus on the notions of probabilistic function, with associated operations of inversion and composition, which are necessary for treating Equation (10).

3. PROBABILITY AS A LOGIC FOR MODELS FORMULATED THROUGH FIRST-ORDER LANGUAGES

3.1. Probabilistic semantic

Let us consider models formulated in *a first order language*, $\boldsymbol{L}$, that is a language whose elementary propositions concern properties of, or relations among, individuals, whilst more complex ones can be formed by combining the elementary ones through logical operators, such as conjunction, $\wedge$, disjunction, $\vee$, or negation, $\neg$ [21]. Such a language is rich enough for our purposes, as it will appear in the following. Once a statement is made, it is of interest to assess its truth or falsity. This is the object of semantic, and the basis of a deterministic semantic, for statements of our interest, has already been presented in previous Section 2.1. The purpose of the proposed theory is to replace a deterministic semantic with a probabilistic one [22].

As we have seen, a *deterministic model*, M_d, may be expressed by a structure, $H = (C, R)$, where $C = A_1 \times A_2 \ldots \times A_p$ is a Cartesian product of sets and $R = \big(R_1, R_2, \ldots, R_q\big)$, where each R_i is a m_i-ary relation on C, expressed in the language $\boldsymbol{L}$. The truth of a generic statement, ϕ, concerning M_d, can be assessed in the following way:

- if ϕ is an elementary proposition, it is true if for some $R_i \in R, \phi \in R_i$,
- if instead it is the combination of elementary propositions, through logical operators, it is true if it satisfies the truth condition of the operators combined with the truth state of the elementary propositions involved.

A *probabilistic model*, M_p, instead is constituted by a finite collection of structures, $E = \{H_1, H_2, \ldots H_M\}$, all associated to the same collection of sets, C, and a probability distribution $P(H_i)$ over E, such that

$$P(E) = \textstyle\sum_{i=1}^{M} P(H_i) = 1. \tag{11}$$

Such structures constitute a set of possible realisations of the same basic underlying structure and are sometimes suggestively called "possible worlds". Then, the probability of any statement ϕ associated to M_p is

$$P(\phi) = P\{H \in E|\phi\} = \sum_{H_i \in E|\phi} P(H_i), \tag{12}$$

where $H \in E|\phi$ denotes a structure where ϕ is true, $\{H \in E|\phi\}$ is the subset of E that includes all the structures in which ϕ is true and the sought probability is the sum of the probabilities of such structures. To apply this approach to measurement, its application to probabilistic m-ary relations and to probabilistic functions must be investigated, with a special focus on probabilistic inversion.

3.2. Probabilistic relations

If $R(u_1, u_2, \ldots u_m)$ is an m-ary relation and E is a finite collection of structures, $H_i(C, R_i)$, where the truth of R can be ascertained, we obtain:

$$P\big(R(a_1, a_2, \ldots a_m)\big) = P\{H \in E|R(a_1, a_2, \ldots a_m)\} = \textstyle\sum_{H_i \in E|R(a_1, a_2, \ldots a_m)} P(H_i)\,. \tag{13}$$

Probabilistic relations were treated in detail in Reference [22] and thus are not pursued further here.

3.3. Probabilistic functions

Considering a function $f: A \rightarrow B$, the associated structure is $H = (A \times B, f)$, and the generic statement $v = f(u)$ denotes a binary relation on $A \times B$ such that $\forall u \in A, \exists v \in B (v = f(u)\,)$, and $\forall u \in A \forall v, z \in B(v = f(u)\ \wedge z = f(u) \rightarrow v = z)$. Let us then consider a finite collection, E, of such structures and an associated probability distribution on E. Then the probability that the above statement holds true for a pair (a, b), $a \in A$, $b \in B$, can be calculated by:

$$P(f(a) = b) = P\{H \in E|f(a) = b\} = \textstyle\sum_{H_i|f(a)=b} P(H_i). \tag{14}$$

3.4. Probabilistic inversion

Consider now the probabilistic inverse to the function f in the previous subsection, i.e., $g: B \rightarrow A$. Let us consider first the possibility of calculating directly the probability associated to each value of g from the knowledge of the corresponding direct function f, through the very definition of inverse function, by establishing the following rule:

$$P(g(b) = a) \propto P\{H \in E|f(a) = b\} = \textstyle\sum_{H_i|f(a)=b} P(H_i). \tag{15}$$

After imposing the *closure condition* $\sum_{u \in A} P(g(b) = u)$=1, we obtain the rule:

$$P(g(b) = a) = \frac{\sum_{H_i|f(a)=b} P(H_i)}{\sum_{u \in A} P(f(u)=b)}. \tag{16}$$

Let us briefly discuss the relationship between probabilistic inversion, as here presented, and the Bayes-Laplace rule. To do that, let now u and v be two variables that denotes *generic* elements of A and B, respectively, and let a and b be two *specific* elements of A and B, respectively. Then we can form the atomic statements $\phi = (u = a)$ and $\psi = (v = b)$, that means, for example, that, in some circumstance, the element $a \in A$

occurred and the element $b \in B$ occurred. Then function f induces a conditional probability measure on $A \times B$, defined by:

$$P(\psi|\phi) = P\big((v = b)|(u = a)\big) = P\big(b = f(a)\big). \quad (17)$$

Then the (inverse) conditional probability $P(\phi|\psi)$, equals the probability of the inverse function, and can be calculated through the Bayes-Laplace rule, with a uniform prior, that is

$$P(\phi|\psi) = P\big((u = a)|(v = b)\big) = \frac{P\big((v = b)|(u = a)\big)}{P\big((v = b)\big)} = P\big(a = g(b)\big) \quad (18)$$

Therefore, in this context, *the Bayes Laplace rule can be interpreted as a procedure for calculating the inverse of a probabilistic function*. Consequently, its use in measurement can be presented just as a step in measurement modelling, as it will be shown in the next section, without taking any commitment to Bayesian statistics, with its philosophical and epistemological implications [23].

3.5. Composition of probabilistic functions

Lastly, let f, g, and h be three probabilistic functions, $f: A \to B$, $g: B \to C$ and $h: A \to C$, where for $u \in A$, $h(u) = g\big(f(u)\big)$. Then the probability of statements concerning h can be assessed through the rule:

$$P\big(w = h(u)\big) = \sum_{v \in B} P\left(w = g(v)\right) P\big(v = f(u)\big) \quad (19)$$

where $w \in C$. Let us now apply the above rules to the probabilistic modelling of measurement processes.

4. PROBABILITY AS A LOGIC FOR MEASUREMENT MODELLING

4.1. Modelling the measurand

In Section 2.2 a deterministic model was developed, based on Equations (6) and (7). Such model implies that empirical relations appearing in it are uncertainty-free. If, instead, the intrinsic uncertainty of the measurand, which basically corresponds to the "definitional uncertainty" in the VIM, needs considering, such model must be turned into probabilistic. This can be done, by applying Equation (13), to Equation (7), which ultimately yields

$$P(u \succcurlyeq v) = P\big(h(u) \geq h(v)\big), \quad (20)$$

as proved in Reference [22] and similar results can be obtained for all the scales of practical interest.

4.2. Modelling the measurement process

The overall modelling of the measurement process has been outlined in Section 2.3, where it was suggested that the overall measurement process can be described by the measurement function $\gamma: A \to X$, characterised by Equation (10). Therefore, a proper structure for the measurement process is

$$M" = (A \times Y \times X, \varphi, f, \gamma). \quad (21)$$

Yet, this description does not include the modelling of the measurand and does not allow to account for the associated *intrinsic* or *definitional* uncertainty, as previously discussed. This is acceptable in practice when such uncertainty is considered negligible. Yet in the general case, models M' and $M"$ must be merged, yielding (for a purely ordinal quantity) the structure:

$$N = (A \times Y \times X, \succcurlyeq, m, \varphi, f, \gamma). \quad (22)$$

As anticipated, this overall model can be interpreted either as deterministic of probabilistic, after interpreting the relations, variable and/or functions involved accordingly. Recalling the previously presented equations, we obtain for a generic probabilistic statement concerning the measurement function γ, in model M':

$$P\big(\hat{x} = \gamma(a)\big) = P\left(\hat{x} = f^{-1}\big(\varphi(a)\big)\right) = \sum_{y} \frac{P(y=f(\hat{x}))}{\sum_w P(y=f(w))} P\big(y = \varphi(a)\big). \quad (23)$$

On the other hand, if we want to account for intrinsic uncertainty also, we should refer to model N and consider m as a probabilistic function as well. Note, in this regard, that the function $\varphi: A \to Y$, only depends (at least ideally) on the way in which the object a realises and manifests the quantity, x, of interest. Let us call it $x_a = m(a)$. Therefore,

$$y = \varphi(a) = f\big(m(a)\big). \quad (24)$$

4.3. A very simple numerical illustrative example

Let us finally illustrate the entire procedure by a very simple numerical example, concerning the (purely ordinal) height of three subjects, call them John (a), Paul (b) and Evelyn (c). Suppose John is definitely taller than the other two, so that $P(a \succ b) = P(a \succ c) = 1.0$. Let instead Paul be almost as tall as Evelyn", with $P(b \sim c) = 0.6$, $P(b \succ c) = 0.1$ and $P(c \succ b) = 0.3$. Then it is easy to check that a proper function $m: \{a, b, c\} \to \{1,2\}$ will have[3]:

$$P(m(a) = 1) = 0.0; P(m(a) = 2) = 1.0;$$
$$P(m(b) = 1) = 0.9; P(m(b) = 2) = 0.1;$$
$$P(m(c) = 1) = 0.7; P(m(c) = 2) = 0.3.$$

Let us now consider the calibration function, $f: X \to X$ and let $X = \{1,2\}$ and the probability of f be such that:

$$P(f(1) = 1) = 0.8; P(f(1) = 2) = 0.2;$$
$$P(f(2) = 1) = 0.1; P(f(2) = 2) = 0.9.$$

Then the probability of the inverse function g is such that:

$$P(g(1) = 1) = 8/9; P(g(1) = 2) = 1/9;$$
$$P(g(2) = 1) = 2/11; P(g(2) = 2) = 9/11.$$

The observation function φ, is obtained by composing f and m, according to Equation (19), which yields:

$$P(\varphi(a) = 1) = 0.10; P(\varphi(a) = 2) = 0.90;$$
$$P(\varphi(b) = 1) = 0.73; P(\varphi(b) = 2) = 0.27;$$
$$P(\varphi(c) = 1) = 0.59; P(\varphi(c) = 2) = 0.41.$$

Lastly, the measurement function γ outcomes from the composition of g with φ, yielding:

$$P(\gamma(a) = 1) = 0.251; P(\gamma(a) = 2) = 0.749;$$
$$P(\gamma(b) = 1) = 0.698; P(\gamma(b) = 2) = 0.302;$$
$$P(\gamma(c) = 1) = 0.599; P(\gamma(c) = 2) = 0.401.$$

5. CONCLUSION

The problem of the interpretation of probability in measurement has been considered and it was suggested to regard

[3] See Reference [22] for additional details of this representational side of the question.

probability theory as a logic for developing probabilistic models. A remarkable feature of this approach is that after modelling measurement through the relations holding among the transformations involved, the model can be treated as either deterministic or probabilistic, depending upon the chosen sematic. Alternative approaches can be considered, such as the fuzzy logic or the possibility theory [9], [10]. All these approaches have their merits and limitations, and the choice may be done depending upon the assumptions made in the development of the model. The logicistic approach here developed may overcome some reservations about probability theory, related to the limits of the frequentistic and the subjectivistic approaches, and may thus contribute to a wider use of the probabilistic approach.

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